Paper

# FDTD analysis of magnetized plasma using an equivalent lumped circuit

Artur Moryc and Wojciech Gwarek

**Abstract— The paper describes a new approach to electromagnetic analysis of magnetized plasma using finite difference time domain (FDTD) method. An equivalent lumped circuit describing an FDTD cell filled with plasma is developed and applied in the analysis. Such a method is proved more effective than previously reported methods. The new approach is verified on a canonical example of known analytical solution.**

***Keywords— FDTD, tensor permittivity, dispersive, anisotropic.***

## 1. Introduction

Finite difference time domain (FDTD) method is one of the most useful methods in modeling of electromagnetic problems [1, 2]. Since its original formulation in the sixties it has been a subject of thousands of publications. One of important problems considered by FDTD researchers is effective analysis of dispersive media. Several approaches to this problem have been reported [3], but the one which seems to be most frequently used now is that of Kuntz and Luebbers [2]. In that approach the frequency dependence of the material properties is transformed into time domain through a convolution. It is shown [2] that an effective recursive updating of the convolution terms is possible. The convolution approach has been also applied to dispersive anisotropic media like magnetized plasma and magnetized ferrites [2, 4, 5]. It proved to be more effective than formerly published approaches like [6]. Recently the authors of this paper have developed an alternative approach to the analysis of magnetized ferrites [7]. A lumped equivalent circuit for ferrite-loaded FDTD cell has been developed and it has been applied to electromagnetic modeling algorithm. It proved simpler and more effective than previously reported algorithms. In this paper we extend application of that approach to the case of magnetized plasma. An equivalent lumped circuit describing an FDTD cell filled with plasma is developed and applied to electromagnetic modeling. Example of application shows perfect agreement with analytical solution of a canonical 1D example. The new approach is shown to be computationally more effective than formerly reported approaches [4, 5, 6] in the case of 2D and 3D circuits.

## 2. Plasma and its RLC models

The magnetized plasma is characterized in frequency domain by tensor permittivity which may be written as:

$$\varepsilon = \begin{pmatrix} \varepsilon_{xx}(\omega) & j\varepsilon_{xy}(\omega) & 0 \\ -j\varepsilon_{yx}(\omega) & \varepsilon_{yy}(\omega) & 0 \\ 0 & 0 & \varepsilon_{zz}(\omega) \end{pmatrix}, \tag{1}$$

$$\varepsilon_{xx}(\omega) = \varepsilon_{yy}(\omega) = 1 - \frac{(\frac{\omega_p}{\omega})^2[1-j\frac{\nu_c}{\omega}]}{[1-j\frac{\nu_c}{\omega}]^2 - (\frac{\omega_b}{\omega})^2}, \tag{2}$$

$$\varepsilon_{yx}(\omega) = \varepsilon_{xy}(\omega) = \frac{(\frac{\omega_p}{\omega})^2\frac{\omega_b}{\omega}}{[1-j\frac{\nu_c}{\omega}]^2 - (\frac{\omega_b}{\omega})^2}, \tag{3}$$

$$\varepsilon_{zz}(\omega) = 1 + \frac{(\omega_p)^2}{\omega(j\nu_c - \omega)}, \tag{4}$$

where $\omega_p$ is plasma frequency, $\omega_b$ is cyclotron frequency (proportional to the static filed $H_0$), and $\nu_c$ is the electron collision frequency, which describes the losses of the plasma medium.

Let us start from considering a 2-dimensional case of a TM wave propagating between two magnetic planes situated at $z=0$ and $z=h$. Such a wave has two E-field components $(E_x, E_y)$ and one H-field component $(H_z)$. We can associate with them magnetic voltage $V = aH_z$ and magnetic current $J = -i_z E$. Under such assumptions the Maxwell equations can be written in form of (5) and(6) dual to the form of TE wave case considered in [8]:

$$\bigtriangledown V(x,y,t) = -L\frac{\partial J(x,y,t)}{\partial t}, \tag{5}$$

$$\bigtriangledown J(x,y,t) = -C\frac{\partial V(x,y,t)}{\partial t}. \tag{6}$$

Using FD discretization of space with cell size $a$ and assuming $h=a$, these equations can be expressed for lossless case in a form:

$$-j\omega C V^{m,n} = I_x^{m+0.5,n} - I_x^{m-0.5,n} + I_y^{m,n+0.5} - I_y^{m,n-0.5}, \tag{7}$$

$$-j\omega L I_x^{m+0.5,n} = V^{m+1,n} - V^{m+1,n}, \tag{8}$$

where $V^{m,n} = aH_z(x=ma, y=na)$, $I_x^{m,n} = -aE_y(x=ma, y=na)$, $I_y^{m,n} = aE_x(x=ma, y=na)$, $C = \varepsilon a$ and $L = \mu a$.

Let us note that $LC$ parameters have unusual meaning, i.e., $L$ is related to electric field and $C$ is related to magnetic field. We start from considering the case of isotropic cold plasma described by isotropic permittivity given by Eq. (4) (as in [3]). Figure 1 presents an equivalent lumped

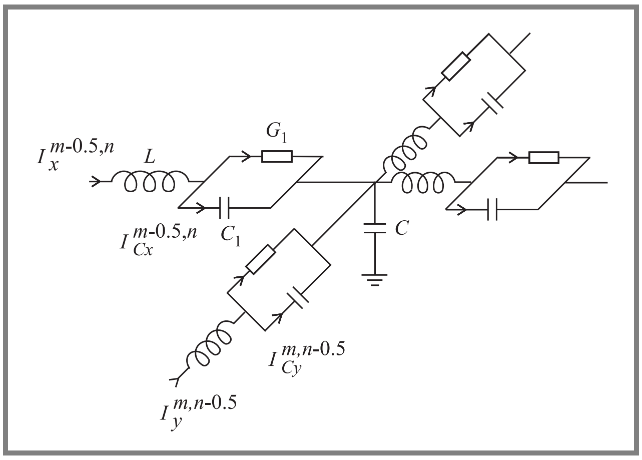


***Fig. 1.*** Lumped equivalent circuit for isotropic plasma.

circuit describing such a medium. Then we switch to the magnetized plasma case. We consider a plasma material in which electric field E is related to displacement vector **D** by permittivity tensor given by Eqs. (1)–(4). Equation (7) does not change while Eq. (8) needs to be modified to the form of:

$$V^{m+1,n} - V^{m,n} = -\mathrm{j}\omega h\varepsilon_0 \left[ (1+\chi) I_x^{m+0.5,n} - \mathrm{j} \frac{\frac{w_b}{w}}{1-\mathrm{j}\frac{\nu_c}{\omega}} \chi I_y^{m+0.5,n} \right], \qquad (9)$$

where c can be expressed as:

$$\chi = \frac{w_p^2 \left(1 - \mathrm{j}\frac{R_1/L_1}{\omega}\right)}{(G_1 R_1 + 1)\omega_b^2 - \omega^2 + \mathrm{j}\omega\left(\frac{G_1}{C_1} + \frac{R_1}{L_1}\right)}, \qquad (10)$$

where $\omega_b^2 = \frac{1}{L_1 C_1}$, $\omega_p^2 = \frac{1}{LC_1}$, $G_1 = C_1 \nu_c$ and $R_1 = L_1 \nu_c$.

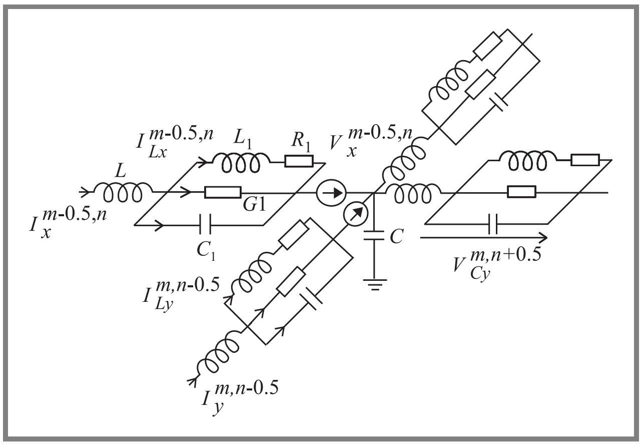


***Fig. 2.*** Lumped equivalent circuit for anisotropic plasma.

Let us note that term (10) corresponds to parallel LC circuit resonating at the angular frequency $\omega_b$ and term $V_{cx}^{m+0.5,n} = -\frac{\mathrm{j}\omega L}{j\omega L_1 + R_1}\chi I_x^{m+0.5,n}$ describes voltage across the parallel LC circuit. The current $I_{Lx}^{m+0.5,n}$ flowing through the inductance arm is proporctional to this voltage multiplied by the admitance of the $L_1 R_1$ conection. Thus we can describe our 2D FDTD cell filled with magnetized plasma by the lumped circuit of Fig. 2 with the source $V_x$ driven by the current $I_{Ly}$.

# 3. Algorithm

The sequence of time-domain updating of FDTD equations for $E_x$ and $E_y$ fields needs to be carefully considered since in the equations containing the effects of off-diagonal anisotropy it is assumed that we should know at the same instant of time $E_x$ to calculate $E_y$ and $E_y$ to calculate $E_x$. The problem has been solved by performing the updates in the following sequence:

1. $E_x^{k+1}$ update based on $E_x^k$, $H_z^{k+0.5}$ and $E_y^k$ components.
2. $E_y^{k+1}$ update based on $E_y^k$, $H_z^{k+0.5}$ and $E_x^{k+1}$ components.
3. Update of $H_z^{k+1.5}$ components.
4. $E_y^{k+2}$ update based on $E_y^{k+1}$, $H_z^{k+1.5}$ and $E_x^{k+1}$ components.
5. $E_x^{k+2}$ update based on $E_x^{k+1}$, $H_z^{k+1.5}$ and $E_y^{k+2}$ components.

Thus the FDTD algorithm based on the equivalent circuit of Fig. 2 proceeds the following way:

$$E_{x,m,n,i}^{k+1} = E_{x,m,n,i}^{k} + \Big(H_{z,m,n+0.5,i}^{k+0.5} - H_{z,m,n-0.5,i}^{k+0.5} - p_{x,m,n,i}^{k+0.5} + F_1\big(e_{y,m-1,n+0.5,i}^{k} + e_{y,m-1,n-0.5,i}^{k} + e_{y,m,n-0.5,i}^{k} + e_{y,m,n+0.5,i}^{k}\big)\Big)F_2, \qquad (11)$$

$$e_{x,m,n,i}^{k+1} = F_3 e_{x,m,n,i}^{k} + F_4 p_{x,m,n,i}^{k+0.5}, \qquad (12)$$

$$p_{x,m,n,i}^{k+1.5} = F_5 p_{x,m,n,i}^{k+0.5} + (E_{x,m,n,i}^{k+1} - e_{x,m,n,i}^{k+1})F_6, \qquad (13)$$

$$E_{y,m,n,i}^{k+1} = E_{y,m,n,i}^{k} + \Big(H_{z,m-0.5,n,i}^{k+0.5} - H_{z,m+0.5,n,i-0.5}^{k+0.5} - p_{y,m,n,i}^{k+0.5} - F_1\big(e_{x,m-0.5,n-1,i}^{k+1} + e_{x,m+0.5,n-1,i}^{k+1} + e_{x,m-0.5,n,i}^{k+1} + e_{x,m+0.5,n,i}^{k+1}\big)\Big)F_2 \qquad (14)$$

$$e_{y,m,n,i}^{k+1} = F_3 e_{y,m,n,i}^{k} + F_4 p_{y,m,n,i}^{k+0.5}, \qquad (15)$$

$$p_{y,m,n,i}^{k+1.5} = F_5 p_{y,m,n,i}^{k+0.5} + (E_{y,m,n,i}^{k+1} - e_{y,m,n,i}^{k+1})F_6. \qquad (16)$$

Where $p_x$, $p_y$, $e_x$, $e_y$ are algorithm varibles corresponding to $V_{cx}$,$V_{cy}$,$I_x$,$I_y$ in Fig. 2, respectively, $F_1 = 0.25L_1\omega_b$, $F_2 = \frac{\Delta t}{C}$, $F_3 = \frac{1-\frac{R_1\Delta t}{2L_1}}{1+\frac{R_1\Delta t}{2L_1}}$, $F_4 = \frac{\Delta t}{L_1}\frac{1}{1+\frac{R_1\Delta t}{2L_1}}$, $F_5 = \frac{1-\frac{G_1\Delta t}{2C_1}}{1+\frac{G_1\Delta t}{2C_1}}$, and $F_6 = \frac{\Delta t}{C_1}\frac{1}{1+\frac{G_1\Delta t}{2C_1}}$.

The algorithm presented here for a 2D can be relatively easily extended to a 3D case. Table 1 presents comparison of computer resources needed for our algorithm and for that of Kunz and Luebbers [2]. Operation count is understood as

Table 1
Comparison of operation count of various methods

| Algorithm | Variables | | Operation count | | | |
|---|---|---|---|---|---|---|
| | 2D | 3D | 2D | | 3D | |
| | | | $V_c > 0$ | $V_c = 0$ | $V_c > 0$ | $V_c = 0$ |
| Our method | 7 | 10 | 37 | 33 | 56 | 52 |
| Kunz and Luebbers [2] | 7 | 10 | 77 | | 96 | |

the number of floating point operations needed per FDTD cell and per iteration. It can be seen that the algorithm presented here is significantly more effective.

## 4. Example

Verification of the accuracy of our algorithm has been conducted on a canonical 1D problem previously considered in [5]. A Gaussian pulse plane wave is normally incident on a longitudinally magnetized plasma layer. The pulse travels through 350 FDTD cells of the total length of 15 mm. The magnetized plasma is placed between cells numbered 200 and 320 and its length is 9 mm. The other cells are filled with air. Both ends of the free space region are terminated by Mur absorbing boundary condition. For these simulations the following parameters of the plasma were assumed:

$$\omega_p = 2\pi \cdot 50 \cdot 10^9 \ \frac{\text{rad}}{\text{s}},$$

$$\omega_b = 3 \cdot 10^{11} \ \frac{\text{rad}}{\text{s}},$$

$$\nu_c = 2 \cdot 10^{10} \ \frac{\text{rad}}{\text{s}}.$$

The $S$ parameters versus frequency for vertical and horizontal polarization were calculated just in front of and behind the plasma. These parameters were used to calculate right-

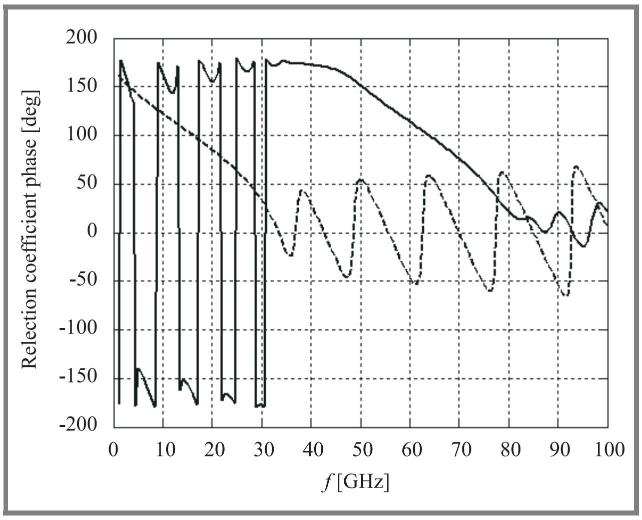


*Fig. 4.* FDTD reflection coefficient phase versus frequency for a plane wave incident on a plasma slab, continuous line corresponds to RCP, dashed line corresponds to LCP.

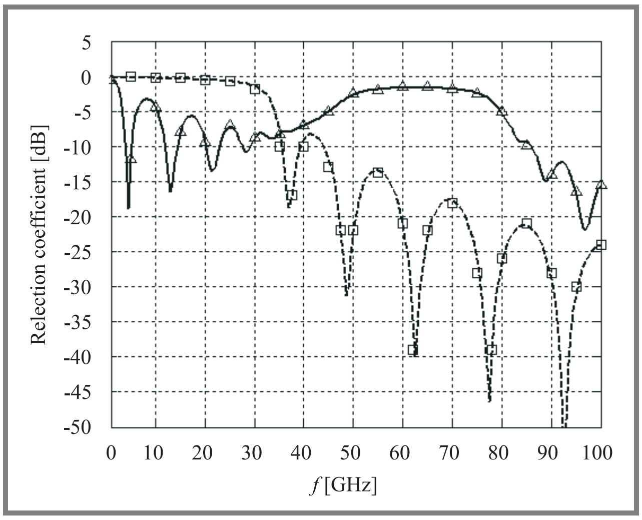


*Fig. 3.* FDTD reflection coefficient magnitude versus frequency for a plane wave incident on a plasma slab, continuous line corresponds to RCP, dashed line corresponds to LCP, triangles and squares the analytical results after [5].

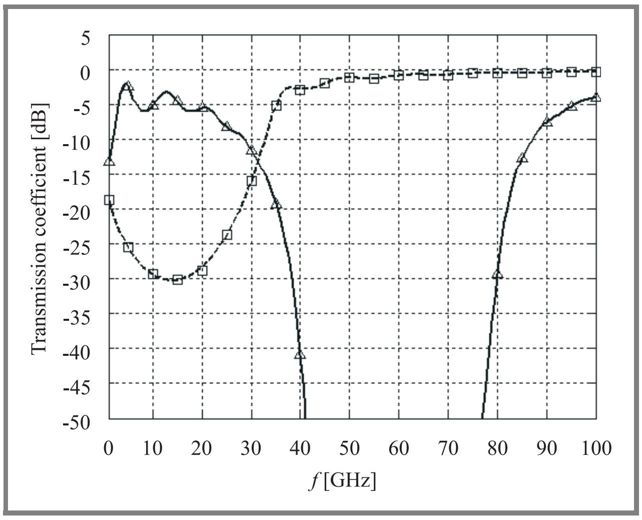


*Fig. 5.* FDTD transmission coefficient magnitude versus frequency for a plane wave incident on a plasma slab, continuous line corresponds to RCP, dashed line corresponds to LCP, triangles and squares the analytical results after [5].

hand and left-hand circularly polarized coefficients according to the formulae:

$$T_{RCP}(\omega) = S_{21x}(\omega) + jS_{21y}(\omega), \quad (17)$$

$$T_{LCP}(\omega) = S_{21x}(\omega) - jS_{21y}(\omega), \quad (18)$$

$$R_{RCP}(\omega) = S_{11x}(\omega) + jS_{11y}(\omega), \quad (19)$$

$$R_{LCP}(\omega) = S_{11x}(\omega) - jS_{11y}(\omega). \quad (20)$$

Figure 3 shows calculated results of reflection coefficients for right-hand and left-hand circular polarizations (denoted RCP and LCP, respectively). Perfect agreement with analytical results after [5] are obtained. Similarly good agreement

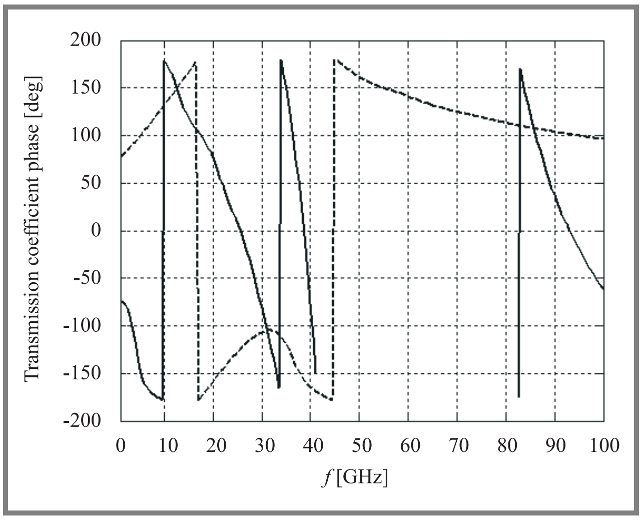


***Fig. 6.*** FDTD transmission coefficient phase versus frequency for a plane wave incident on a plasma slab, continuous line corresponds to RCP (from 40 GHz to 80 GHz data are not presented due to phase uncertainties when the attenuation drops below –50 dB), dashed line corresponds to LCP.

with [5] has been obtained for the transmission coefficients (shown in Fig. 5) and for phase relations for both coefficients (shown in Figs. 4 and 6).

## 5. Conclusions

In this paper a new approach to FDTD analysis of magnetized plasma has been presented. It has been verified on a canonical example with known analytical solution. It proved to be more effective than previously reported approaches. At the same time it has been found robust and reliable in practical simulations of 2D and 3D cases.

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**Wojciech Gwarek** – for biography, see this issue, p. 28.