Paper

Possibilistic approach to Bayes decisions

Olgierd Hryniewicz

Abstract — The decision problems are considered when the prior probabilistic information about the state of nature and decision maker's utility function are imprecisely defined. In such a case the risks (or the expected utility) of considered decisions are also imprecisely defined. We propose two-step procedure for finding the optimal decision. First, we order possible decisions using the λ-average ranking method by Campos and Gonzalez [1]. Then we use possibilistic possibility of dominance and necessity of strict dominance indices proposed by Dubois and Prade [3] for the comparison of consequences of the most promising solutions.

Keywords* — *optimal decisions, imprecise information, fuzzy risks, possibility indices.

1. Introduction

In decision making we deal with uncertainties related to an unknown state of nature. The behaviour of a decision maker may be described as a kind of game between him and a fictitious player who may be called "nature" or "chance". Decisions made by a decision maker are rational if they are derived from his knowledge about nature's behaviour and the knowledge of the consequences of his decisions. Mathematical theories of decision making are known for more than fifty years and are based on probabilistic models of nature's behaviour and utility functions. Their basic ideas and main results were published in a famous book by Raiffa and Schlaifer [6] that has been recently republished by J. Wiley & Sons. In the classical models of decision making it is assumed that the decision maker knows the joint probability distribution of all possible states of the nature and all possible results of experiments which provide him with some knowledge about the actual state of the nature. Moreover, it is assumed that there exists a precisely defined utility function which assigns decision maker's utility related to all possible pairs: decision and state of the nature. These premises have been recently relaxed by assuming that some parameters of decision models may be defined only with a certain degree of precision. As a consequence of such more general assumptions we arrive at mathematical models of imprecise risks.

In this paper we present some results obtained under the assumption of the existence of imprecisely defined risks. In Section 2 we present a mathematical model of decision making in the presence of imprecisely defined probabilistic prior information about the possible states of the nature and imprecisely defined utility functions. A lack of the precision we describe in the language of the fuzzy sets theory. We propose to find the best decisions by the defuzzification of imprecisely defined expected risks. For this purpose we propose the use of the defuzzification method proposed by Campos and Gonzalez [1]. This method allows the user to take into account his attitude, i.e. his level of optimism (or pessimism). In Section 3 we propose a possibilistic method for the comparison of different decisions. By applying this method we provide the user with addition information about the real differences between the consequences of his decisions. In this comparison we take into account the impact of imprecise input information on the decision making.

2. Mathematical model and the choice of optimal decisions

There exist different methods for modelling decisions. In this paper we adopt the approach described in a general form by Raiffa and Schlaifer [6]. The model proposed by Raiffa and Schlaifer consists of two parts: one part is dedicated to the choice of the final decision, and the second part is dedicated to the choice of the experiment whose ultimate goal is to provide the decision maker with some information about the actual state of nature. According to this model the decision maker can specify the following data defining his decision problem:

1. Space of terminal decisions (acts): $A = \{a\}$.
2. State space: $\Theta = \{\theta\}$.
3. Family of experiments: $E = \{e\}$.
4. Sample space: $Z = \{z\}$.
5. Utility function: $u(\cdot,\cdot,\cdot,\cdot,)$ on $E \times Z \times A \times \Theta$.

The decision maker evaluates an utility $u(e,z,a,\theta)$ of making a particular experiment e, obtaining the result of this experiment z, taking a decision a in the case when the true state of nature is θ. In order to find appropriate (hopefully optimal) decisions the decision maker has also to specify a joint probability measure $P_{\theta,z}(\cdot,\cdot|e)$ for a Cartesian product $\Theta \times Z$. The knowledge of this probability measure means that we know the joint probability distribution of observation z in an experiment e when the *random* state of nature is described by θ. Knowing this joint probability distribution we can calculate some important marginal and conditional probability distributions. In particular, for a given experiment e we are usually interested in three distributions:

1. The marginal distribution on the state space Θ describing our *prior* information about possible states of nature. We assume that this distribution does not depend on e.

2. The conditional distribution on the sample space Z for a given state of nature θ.

3. The conditional distribution on the state space Θ for a given result of the experiment z describing our *posterior* information about possible states of nature.

Note, that we may know only these particular distributions as their knowledge is equivalent to the knowledge of the joint probability distribution on $\Theta \times Z$.

Let us consider the simplest case of the general model when there is no experiment e. In such a case the only information we need is the probability distribution $\pi(\theta)$ defined on the state space Θ. We call this distribution *the prior distribution* of the parameter (parameters) describing the unknown state of nature. If we know the utility function $u(a,\theta)$ defined on $A \times \Theta$ we may calculate *the expected utility* assigned to a particular action (decision) a from the simple formula

$$u(a) = \int_{\Theta} u(a,\theta)\,\pi(\theta)\,\mathrm{d}\theta\,. \tag{1}$$

If we use *a loss function* $L(a,\theta)$ for the description of potential consequences of taking decision a we may calculate *the expected* loss (usually called *a risk*) from an equivalent formula

$$\rho(a) = \int_{\Theta} L(a,\theta)\,\pi(\theta)\,\mathrm{d}\theta\,. \tag{2}$$

Having the expected utilities for *all* possible decisions we can find the optimal one which is related to the maximal expected utility (or the minimal risk). This procedure is in principle very simple. However, in many practical cases (when the number of possible decisions is sufficiently large) it may require the use of sophisticated optimisation methods.

When the decision maker has an additional information about the state of nature in a form of observations $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ of a random vector described by a probability distribution $f(\mathbf{z},\theta)$ we may calculate *the expected utility* assigned to a particular action (decision) a from a formula

$$u(a,\mathbf{z}) = \int_{\Theta} u(a,\theta)\,g(\theta|\mathbf{z})\,\mathrm{d}\theta\,, \tag{3}$$

where

$$g(\theta|\mathbf{z}) = \frac{f(\mathbf{z}|\theta)\,\pi(\theta)}{\int_{\Theta} f(\mathbf{z}|\theta)\,\pi(\theta)\,\mathrm{d}\theta} \tag{4}$$

is the posterior distribution of the parameter θ which describes the state of nature. In such a case the expected utility attributed to each decision is calculated from

$$u(a|\mathbf{z}) = \int_{\Theta} u(a,\theta)\,g(\theta|\mathbf{z})\,\mathrm{d}\theta\,, \tag{5}$$

and the respective risk from the formula

$$\rho(a|\mathbf{z}) = \int_{\Theta} L(a,\theta)\,g(\theta|\mathbf{z})\,\mathrm{d}\theta\,. \tag{6}$$

The procedure for finding the optimal decision is exactly the same as in the case described previously.

Suppose now that the prior distribution $\pi(\theta)$ and the loss (or utility) $L(a,\theta)$ are functions of parameters ζ and ψ, respectively, and that these parameters are known only imprecisely. Let us assume that our imprecise knowledge about possible values of ζ and ψ is represented by fuzzy sets $\widetilde{\zeta}$ and $\widetilde{\psi}$, respectively. A fuzzy set $\widetilde{X}$ is defined using the membership function $\mu_{\widetilde{X}}(x)$ which in the considered context of this paper describes the grade of possibility that a fuzzy parameter, say $\widetilde{X}$, has a specified value of x. Each fuzzy set may be also represented by its α-cuts defined as ordinary sets

$$X^{\alpha} = \left\{x \in R : \mu_{\widetilde{X}}(x) \geq \alpha\right\}, \quad 0 \leq \alpha \leq 1\,. \tag{7}$$

From the representation theorem for fuzzy sets we know that each membership function may be equivalently expressed as

$$\mu_{\widetilde{X}}(x) = \sup\left\{\alpha I_{\widetilde{X}^{\alpha}}(x) : \alpha \in [0,1]\right\}. \tag{8}$$

Now let us assume that imprecisely known parameters ζ and ψ are represented by their α-cuts, and that these α-cuts are given in a form of closed intervals $\left[\zeta_L^{\alpha}, \zeta_U^{\alpha}\right]$ and $\left[\psi_L^{\alpha}, \psi_U^{\alpha}\right]$, respectively. The knowledge of these α-cuts let us calculate fuzzy equivalents of the expected utility or the expected loss (risk). To make the presentation simple we assume that decision are based exclusively on the knowledge of the prior distribution $\pi(\theta)$ and the loss function $L(a,\theta)$. As these functions are the functions of imprecise fuzzy parameters, they are also fuzzy, and may be denoted as $\widetilde{\pi}(\theta;\widetilde{\zeta})$ and $\widetilde{L}(a,\theta;\widetilde{\psi})$, respectively.

Now, let us rewrite formula (2) as

$$\widetilde{\rho}(a) = \int_{\Theta} \widetilde{L}(a,\theta;\widetilde{\zeta})\widetilde{\pi}(\theta;\widetilde{\psi})\mathrm{d}\theta\,. \tag{9}$$

The risk calculated from formula (9) is now an imprecisely defined *fuzzy number* whose membership function may be calculated using Zadeh's extension principle (see Klir and Yuan [5], or any other textbook on fuzzy sets for a reference). It is easy to show that the fuzzy risk $\widetilde{\rho}(a)$ is now represented by its α-cuts $\left[\rho_L^\alpha, \rho_U^\alpha\right]$, where

$$\rho_L^\alpha = \inf_{\substack{\zeta \in \left[\zeta_L^\alpha, \zeta_U^\alpha\right] \\ \psi \in \left[\psi_L^\alpha, \psi_U^\alpha\right]}} \widetilde{\rho}(a) \tag{10}$$

and

$$\rho_U^\alpha = \sup_{\substack{\zeta \in \left[\zeta_L^\alpha, \zeta_U^\alpha\right] \\ \psi \in \left[\psi_L^\alpha, \psi_U^\alpha\right]}} \widetilde{\rho}(a). \tag{11}$$

Thus, for every possible decision a we may find a fuzzy risk $\widetilde{\rho}(a)$ or a fuzzy expected utility $\widetilde{u}(a)$ which may be calculated in the same way. Moreover, if there exists an additional information in the form of observations $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ we may use exactly the same procedure in order to fuzzify the expected utility given by formula (5) and the risk given by formula (6). Note however, that in this case the respective calculations (especially for the fuzzy posterior distribution) may be much more complicated.

In contrast to the non-fuzzy (crisp) case the univocal optimal solution of the decision problem for imprecisely defined input parameters does not exist. It stems from the fact that fuzzy sets are not naturally ordered. Thus, in general, it is not possible to indicate the decision with lowest risk (or the highest expected utility). In order to do this we must apply one of the many proposed in literature ranking methods.

There are many methods for ranking fuzzy numbers that are based on different defuzzification methods. Gil and Lopez-Diaz [4] have noticed that the λ-average ranking method proposed by Campos and Gonzalez [1] is especially useful in decision making. Let $\widetilde{X}$ be a fuzzy number (fuzzy set) described by the set of its α-cuts $\left[X_L^\alpha, X_U^\alpha\right]$, and S be an additive measure on $[0,1]$. Moreover, assume that the support of $\widetilde{X}$ is a closed interval. The λ-average value of such a fuzzy number $\widetilde{X}$ is defined by Campos and Gonzalez [1] as

$$V_S^\lambda(\widetilde{X}) = \int_0^1 \left[\lambda X_U^\alpha + (1-\lambda) X_L^\alpha\right] \mathrm{d}S(\alpha), \quad \lambda \in [0,1]. \tag{12}$$

In the case of continuous membership functions the integral in formula (12) is calculated with respect to $\mathrm{d}\alpha$. Thus, the λ-average value of $\widetilde{X}$ can be viewed as its defuzzified value.

The parameter λ in (12) is a subjective degree of decision maker's optimism (pessimism). In the case of fuzzy risks $\lambda = 0$ reflects his highest optimism as the minimal values of all α-cuts (representing the lowest possible risks) are taken into consideration. On the other hand, by taking $\lambda = 1$ the decision maker demonstrates his total pessimism, as only the maximal values of all α-cuts (representing the highest possible risks) are considered. In the case of fuzzy expected utilities the situation is reversed, i.e. $\lambda = 1$ represents decision maker's optimism, and $\lambda = 0$ reflects his total pessimism. If the decision maker takes $\lambda = 0.5$ his attitude may described as neutral. Thus, by varying the value of λ the decision maker is able to take into account the level of his optimism (pessimism) which may arise e.g. from having some additional information that has not been reflected in the prior distribution. Some interesting features of the λ-average ranking method have been discussed in Gil and Lopez-Diaz [4].

Having a simple ranking method given by formula (12) we may calculate defuzzified values of fuzzy risks (expected utilities) related to all considered decisions. The optimal decision has the lowest defuzzified risk (or the highest defuzzified expected utility). Moreover, the decision maker can order all considered decisions with respect to their risks (or expected utilities).

3. Possibilistic analysis of optimal decisions

The procedure described in the previous section allows the decision maker to find the optimal decision. It has to be noted, however, that the ranking method gives only a partial information about the differences between competitive decisions. Therefore, we claim that it is necessary to perform an additional analysis that provides the decision maker with an additional information about the considered decisions. Such an analysis is especially interesting when the consequences of different decisions are similar, and when other decision maker's preferences, not reflected in the optimisation model, exist. To analyse the consequences of different decisions we propose to use the methodology known from the theory of possibility, namely the *possibility of dominance* and *necessity of strict dominance* indices proposed by Dubois and Prade [3].

For two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ the *possibility of dominance (PD)* index is calculated from the formula

$$PD = Poss(\widetilde{A} \geq \widetilde{B}) = \sup_{x,y:x\geq y} \min\left\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(y)\right\}. \tag{13}$$

The *PD* index gives the measure of *possibility* that the fuzzy number $\widetilde{A}$ is not smaller than the fuzzy number $\widetilde{B}$. Positive value of this index tells the decision maker that there exists even slightly evidence that the relation $\widetilde{A} \geq \widetilde{B}$ is true.

The degree of *conviction* that the relation $\widetilde{A} > \widetilde{B}$ is true is reflected by the *necessity of strict dominance (NSD)* index defined as

$$NSD = Ness(\widetilde{A} > \widetilde{B}) =$$

$$= 1 - \sup_{x,y:x\leq y} \min\left\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(y)\right\} = 1 - Poss(\widetilde{B} \geq \widetilde{A}). \tag{14}$$

The *NSD* index gives the measure of *necessity* that the fuzzy number $\widetilde{A}$ greater than the fuzzy number $\widetilde{B}$. Positive value of this index tells the decision maker that there exists rather strong evidence that the relation $\widetilde{A} > \widetilde{B}$ is true.

According to Cutell and Montero [2] we may use the *PD* and *NSD* indices to evaluate mutual relationship between two considered decisions. Let us describe the evaluation procedure for two decisions a_1 and a_2 with associated fuzzy risks $\widetilde{\rho}(a_1)$ and $\widetilde{\rho}(a_2)$, respectively. The value of $NSD = Nec\big(\widetilde{\rho}(a_1) > \widetilde{\rho}(a_2)\big)$ indicates that extend decision a_1 is inferior in comparison to decision a_2. On the other hand, $1-PD = 1-Poss\big(\widetilde{\rho}(a_1) \geq \widetilde{\rho}(a_2)\big)$ indicates that extend decision a_1 might be considered superior in comparison to decision a_2. If instead of fuzzy risks we compare fuzzy expected utilities the conclusions are reversed, i.e. the value of $NSD - Nec\big(\widetilde{u}\,(a_1) > \widetilde{u}\,(a_2)\big)$ indicates that extend decision a_1 is superior in comparison to decision a_2, etc. The value of $PD - NSD$ may be viewed upon as the measure of *indifference* between the consequences of the considered decisions.

If the decision maker has the ordered sequence of his possible decisions he should always consider a possibility of performing pairwise comparisons between the best two (or more) competitive solutions. High values of the indifference indices reveal that the consequences of considered decisions are rather insignificant due to the lack of precision of the optimisation model. In such a case the decision maker may use some additional criteria for choosing an appropriate decision. This is also the signal that it is advisable to make the optimisation model more precise.

4. Decisions with two possible outcomes – a numerical example

Let us consider the simplest situation when each action from a set of alternatives $\{a_1,\ldots,a_M\}$ leads to two possible outcomes $w^{(m)}$, $m = 1,\ldots,M$ and $v^{(m)}$, $m = 1,\ldots,M$, respectively. The outcome $w^{(m)}$ appears with probability $p^{(m)}$, $m = 1,\ldots,M$, and the outcome $v^{(m)}$ appears with probability $1-p^{(m)}$. Suppose that the expected outcome is equivalent to the expected utility. Thus the expected utility associated with the action a_m is given by

$$u^{(m)} = p^{(m)}w^{(m)} + \big(1-p^{(m)}\big)v^{(m)}, \quad m = 1,\ldots,M. \qquad (15)$$

In this way, the optimal action is a such one which maximises Eq. (15) when the outcomes are given in terms of profits or minimises Eq. (15) when outcomes are expressed in terms of losses.

Let us assume that all information about the outcomes and respective probabilities are imprecise and are given by fuzzy numbers described by a trapezoidal membership functions. In general, any trapezoidal membership function of a fuzzy number $\widetilde{X} = \widetilde{X}(x_l, x_{0,l}, x_{0,r}, x_r)$ is described by the following formula:

$$\mu_{\widetilde{X}}(x) = \begin{cases} 0 & x \leq x_l \\ \dfrac{x-x_l}{x_{0,l}-x_l} & x_l < x \leq x_{0,l} \\ 1 & x_{0,l} < x \leq x_{0,r} \\ \dfrac{x_r-x}{x_r-x_{0,r}} & x_{0,r} < x \leq x_r \\ 0 & x > x_r \end{cases}. \qquad (16)$$

The α-cuts of the fuzzy number described by the membership function given by formula (16) have the following form: $\big(x_l + \alpha(x_{0,l}-x_l), x_r - \alpha(x_r - x_{0,r})\big)$.

Denote by $\widetilde{w}^{(m)}$, $m = 1,\ldots,M$, and $\widetilde{v}^{(m)}$, $m = 1,\ldots,M$ the fuzzy counterparts of the crisp outcomes $w^{(m)}$ and $v^{(m)}$, respectively. Moreover, let $\widetilde{p}^{(m)}$, $m = 1,\ldots,M$ be the fuzzy counterpart of the crisp probability $p^{(m)}$. Assume now, that for each α-cut we have $w^{(m)}_{0,l} > v^{(m)}_{0,r}$. It means that despite their imprecision both possible outcomes are separated. When this assumption does not hold we have either to assume that the outcomes are interactive in a special way or to assume that they are indistinguishable to some extent. In both cases, this leads to severe complication of the optimisation procedure.

Now, we can define a fuzzy expected utility as follows

$$\widetilde{u}^{(m)} = \widetilde{p}^{(m)}\widetilde{w}^{(m)} + (1-\widetilde{p}^{(m)})\widetilde{v}^{(m)}, \; m = 1,\ldots,M. \qquad (17)$$

Using the extension principle of Zadeh we can find the membership function of the fuzzy expected utility $\widetilde{u}^{(m)}$, $m = 1,\ldots,M$. In further calculations in order to simplify the notation we omit the upper index (m) that indicates the undertaken action. Denote by $\big(u_l(\alpha), u_r(\alpha)\big)$ the α-cut of $\widetilde{u}$. By simple calculations we can show that

$$u_l(\alpha) = p_l w_l + (1-p_l)v_l + \alpha\big[(p_{0,l}-p_l)w_l + \\ + p_l(w_{0,l}-w_l) - (p_{0,l}-p_l)v_l + (1-p_l)(v_{0,l}-v_l)\big] + \\ + \alpha^2\big[(p_{0,l}-p_l)(w_{0,l}-w_l) - (p_{0,l}-p_l)(v_{0,l}-v_l)\big] \quad (18)$$

and

$$u_r(\alpha) = p_r w_r + (1-p_r)v_r + \alpha\big[(p_r-p_{0,r})v_r + \\ -(1-p_r)(v_r - v_{r,0}) - p_r(w_r - w_{0,r}) + (p_r - p_{0,r})w_r\big] + \\ + \alpha^2\big[(p_r-p_{0,r})(w_r-w_{0,r}) - (p_r-p_{0,r})(v_r-v_{0,r})\big]. \quad (19)$$

The λ-average value of $\widetilde{u}$ calculated from formula (12) is now given by

$$V^\lambda(\widetilde{u}) = \lambda\Big[\frac{1}{2}(v_r+v_{0,r}) + \frac{1}{3}\big(p_r(w_r-v_r) + \\ + p_{0,r}(w_{0,r}-v_{0,r})\big) + \frac{1}{6}\big(p_r(w_{0,r}-v_{0,r}) + p_{0,r}(w_r-v_r)\big)\Big] + \\ + (1-\lambda)\Big[\frac{1}{2}(v_l+v_{0,l}) + \frac{1}{3}\big(p_l(w_l-v_l) + p_{0,l}(w_{0,l}-v_{0,l})\big) + \\ + \frac{1}{6}\big(p_l(w_{0,l}-v_{0,l}) + p_{0,l}(w_l-v_l)\big)\Big]. \qquad (20)$$

Having λ-average values of the fuzzy expected utilities for all considered actions we can find the optimal one that has the maximal value of $V^{\lambda}\left(\widetilde{u}^{(m)}\right)$.

Let us assume that all considered actions are numbered in such a way that $V^{\lambda}\left(\widetilde{u}^{(1)}\right) \geq V^{\lambda}\left(\widetilde{u}^{(2)}\right) \geq \ldots V^{\lambda}\left(\widetilde{u}^{(M)}\right)$. The next step of the possibilistic analysis consists in the comparison of fuzzy expected utilities $\widetilde{u}^{(1)}$ and $\widetilde{u}^{(2)}$. The analysis of Eq. (14) shows that $NSD\left(\widetilde{u}^{(1)} > \widetilde{u}^{(2)}\right) > 0$ if the relation holds $u_r^{(2)}(1) < u_l^{(1)}(1)$. In such a case $NSD\left(\widetilde{u}^{(1)} > \widetilde{u}^{(2)}\right) = 1 - \alpha^*$, where α^* is the solution of the equation

$$u_r^{(2)}(\alpha) = u_l^{(1)}(\alpha)\,. \tag{21}$$

Let

$$x_1 = p_l w_l + (1 - p_l) v_l\,, \tag{22}$$

$$x_2 = (p_{0,l} - p_l) w_l + p_l (w_{0,l} - w_l) + -(p_{0,l} - p_l) v_l + (1 - p_l)(v_{0,l} - v_l)\,, \tag{23}$$

$$x_3 = (p_{0,l} - p_l)(w_{0,l} - w_l) - (p_{0,l} - p_l)(v_{0,l} - v_l)\,, \tag{24}$$

$$y_1 = p_r w_r + (1 - p_r) v_r\,, \tag{25}$$

$$y_2 = (p_r - p_{0,r}) v_r - (1 - p_r)(v_r - v_{r,0}) + -p_r (w_r - w_{0,r}) + (p_r - p_{0,r}) w_r\,, \tag{26}$$

$$y_3 = (p_r - p_{0,r})(w_r - w_{0,r}) - (p_r - p_{0,r})\,, \tag{27}$$

$$A_1 = x_1 - y_1\,, \tag{28}$$

$$A_2 = x_2 - y_2\,, \tag{29}$$

$$A_3 = x_3 - y_3\,. \tag{30}$$

Hence, the solution of Eq. (21) is given by

$$\alpha^* = \begin{cases} \dfrac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_3} & \text{if} \quad A_3 \neq 0 \\ -A_1/A_2 & \text{if} \quad a_3 = 0 \end{cases}\,. \tag{31}$$

To illustrate these theoretical results let us consider a numerical example. Suppose, that there are four possible actions described by the following sets of their fuzzy parameters:

- action a_1:
  $\widetilde{p}^{(1)} = \widetilde{p}^{(1)}(0.2; 0.25; 0.3; 0.35)$,
  $\widetilde{w}^{(1)} = \widetilde{w}^{(1)}(80; 90; 100; 110)$,
  $\widetilde{v}^{(1)} = \widetilde{v}^{(1)}(20; 25; 30; 35)$;
- action a_2:
  $\widetilde{p}^{(2)} = \widetilde{p}^{(2)}(0.2; 0.25; 0.25; 0.25)$,
  $\widetilde{w}^{(2)} = \widetilde{w}^{(2)}(60; 70; 80; 90)$,
  $\widetilde{v}^{(2)} = \widetilde{v}^{(2)}(15; 20; 20; 25)$;
- action a_3:
  $\widetilde{p}^{(3)} = \widetilde{p}^{(3)}(0.2; 0.25; 0.25; 0.3)$,
  $\widetilde{w}^{(3)} = \widetilde{w}^{(3)}(60; 70; 80; 90)$,
  $\widetilde{v}^{(3)} = \widetilde{v}^{(3)}(-10; 20; 20; 25)$;
- action a_4:
  $\widetilde{p}^{(4)} = \widetilde{p}^{(4)}(0.2; 0.2; 0.2; 0.4)$,
  $\widetilde{w}^{(4)} = \widetilde{w}^{(4)}(30; 60; 60; 70)$,
  $\widetilde{v}^{(4)} = \widetilde{v}^{(4)}(-10; 0; 10; 20)$.

The expected utilities associated with each action are given as fuzzy numbers whose λ-averages calculated according to Eq. (20) are the following (for $\lambda = 0.5$, i.e. for a neutral decision maker):

$$V^{\lambda}\left(\widetilde{u}^{(1)}\right) = 46.33; \qquad V^{\lambda}\left(\widetilde{u}^{(2)}\right) = 33.17;$$
$$V^{\lambda}\left(\widetilde{u}^{(3)}\right) = 29.06; \qquad V^{\lambda}\left(\widetilde{u}^{(4)}\right) = 17.5\,.$$

Thus, action a_1 is visibly better than the others. However, if we compare the fuzzy utility of a_1 with the fuzzy utility of the second best action a_2 we arrive at the following results. For a_1 from Eqs. (21)–(24) we get: $x_1 = 32$, $x_2 = 9$, $x_3 = 0.25$, and for a_2 from Eqs. (25)–(27) we get: $y_1 = = 41.25$, $y_2 = -6.25$, $y_3 = 0$. Hence, from Eqs. (28)–(30) we get: $A_1 = -9.25$, $A_2 = 15.25$, $A_3 = 0.25$. Thus, from Eq. (31) we obtain $\alpha^* = 0.6$, and the necessity of strict dominance index is the following $NSD\left(\widetilde{u}^{(1)} > \widetilde{u}^{(2)}\right) = 0.4$. It means that there exists only limited necessity that a_1 is better than a_2, and – to some extent – their results are indistinguishable. This is especially true, when the parameters of the decision model come from different sources.

5. Conclusions

In the paper we present a generalisation of a classical Bayes decision model. In this generalised model we assume that all input parameters describing prior probabilities, costs, and statistical data may be expressed in an imprecise way. If we apply a fuzzy description of those vague data we arrive at fuzzy risks or fuzzy expected utilities associated with each possible action (decision). Unfortunately, a method for an unique ordering of fuzzy numbers does not exist. Therefore, we propose to use the defuzzification method of Campos and Gonzalez [1] in order to find two possibly best actions. Imprecise consequences of these decisions we compare using possibility and necessity indices. This approach gives us a better insight in the process of decision making. We illustrate the proposed procedure with a numerical example when each action (decision) may result with two possible outcomes.

References

[1] L. M. Campos and A. Gonzalez, "A subjective approach for ranking fuzzy numbers", *Fuzzy Sets Syst.*, vol. 29, pp. 145–153, 1989.

[2] V. Cutell and J. Montero, "An extension of the axioms of utility theory based on fuzzy rationality measures" in *Preference and Decisions under Incomplete Knowledge*, J. Fodor, P. De Baets, and P. Perny, Eds. Heidelberg: Physica-Verlag, 1999, pp. 33–50.

[3] D. Dubois and H. Prade, "Ranking fuzzy numbers in the setting of possibility theory", *Inform. Sci.*, vol. 30, pp. 184–244, 1983.

[4] M. A. Gil and M. Lopez-Diaz, "Fundamentals and Bayesian analyses of decision problems with fuzzy-valued utilities", *Int. J. Approx. Reason.*, vol. 15, pp. 203–224, 1996.

[5] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic. Theory and Applications*. New York: Prentice-Hall, 1995.

[6] H. Raiffa and R. Schlaifer, *Applied Statistical Decision Theory.* New York: Wiley, 2000.

Olgierd Hryniewicz is a Professor at the Polish Academy of Sciences, Director of the Systems Research Institute of PAS, and Head of the Laboratory of Stochastic Methods. He received his M.Sc. degree from Warsaw University of Technology (1970). The Ph.D. degree he received from the Institute of Management and Automatic Control of PAS (1976), and the D.Sc. degree from Cracow Academy of Economics (1986). Since 1996 he is a Professor at the Polish Academy of Sciences. He published over 150 books, papers, and reports on reliability, quality control and decision support systems. He is also a Professor of Warsaw School of Information Technology and Management.

e-mail: hryniewi@ibspan.waw.pl
Systems Research Institute
Polish Academy of Sciences
Newelska st 6
01-447 Warsaw, Poland